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$\pi^0 \rightarrow \gamma\gamma$ in the Spinor Strong Interaction Theory

F. C. Hoh[1]

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An expression for the decay rate $\Gamma(\pi^0 \rightarrow \gamma\gamma)$ has been derived in the framework of the spinor strong interaction theory, a first-principles strong interaction theory proposed some years ago as an alternative to low-energy QCD. The starting point is the SO(3) gauge-invariant action for two quark mesons which has been successful in accounting for confinement, $\pi^+ \rightarrow \mu^+ \nu$, $e^+\nu$, and $\pi^0\, e^+\nu$, nonexistence of the Higgs boson, and other low-energy mesonic phenomena. The quasi-four-quark meson equations developed for the decay of a vector meson into two pseudoscalar mesons V $\rightarrow$ PP has been taken over here to apply to P(π^0) $\rightarrow$ VV($\rho^+\rho^-$) $\rightarrow$ $\gamma\gamma$(plus π^+ and π^- which annihilate each other). This mechanism in principle agrees with that of the assumption of vector meson dominance in the literature. It, together with the effect of form factors, arises naturally in the formalism and need not be assumed. Equations for the perturbed vector meson wave functions cannot be simply solved and an assumption has been made to obtain an estimate of their magnitude. Together with a constant associated with the strong coupling obtained earlier from V(φ) $\rightarrow$ PP(K^+K^-), the estimated decay rate is 19.2 eV, in order-of-magnitude agreement with data (7.74 eV).

1. INTRODUCTION

The mechanism of $\pi^0 \rightarrow \gamma\gamma$ is still not known with certainty despite the numerous more or less successful predictions put forward during the past half century. This decay is of central importance in hadron interactions. The rates of some rare decays like $\pi^0 \rightarrow e^+e^-$ also depend upon this mechanism. All these predictions are based upon phenomenological and local models. This is due to the strong interaction character of this decay and the fact that QCD, the current mainstream first-principles strong interaction theory, is incapable of accounting for it, as for most other low-energy phenomena.

There are many different models employing different decay mechanisms. These are largely of the following three types. Five decades ago, Steinberger

[1] Dragarbrunnsg. 9B, 75332 Uppsala, Sweden; e-mail (at The Svedberg Laboratory, Uppsala University): hoh@tsl.uu.se.

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[1] first predicted the rate to be 13.8 eV, remarkbly close to the present data of 7.74 eV. The mechanism is $\pi^0 \to \bar{p}p$, where p stands for proton, whereby each p emits a photon prior to their annihilation. This idea has, with modifications, persisted through the decades. Then came the vector meson dominance (VMD) models [2–5], in which the intermediate state is assumed to be dominated by vector mesons. Finally, the decay has been treated in the framework of current algebra [6] and the intermediate state consists of quarks. There are variations in each type; an example is the incorporation of the Bethe–Salpeter formalism in the quark model [7]. A brief discussion of these models has been given [5].

Since these models have mainly been tailored for this type of decay, they have very narrow application angles. Thus, these models have largely no relation to other low-energy phenomena, such as those mentioned in (i)–(iv) in the next paragraph. Without a sufficient number of such relations, the correctness of any decay mechanism implied by these models is uncertain, inasmuch as there is only one data point, the decay rate 7.74 eV, for the models to account for.

The purpose of this paper is to address this situation and to present a treatment of $\pi^0 \to \gamma\gamma$ within the framework of the spinor strong interaction theory. This theory is a first-principles and nonlocal theory proposed some years ago [8] as an alternative to low-energy QCD. Together with a series of subsequent papers [9–14], this theory has accounted for (i) $\pi^+ \to \mu^+\nu$ and $\pi^0 e^+\nu$, (ii) confinement, ground-state meson spectra, and classification of mesons, (iii) nonexistence of ground-state scalar and axial vector mesons, pseudoscalar isosinglets, and Higgs boson, and (iv) some strong and electromagnetic decays of vector mesons. At high energies, this theory possesses features resembling those of QCD, but no work has been done on this.

The present treatment relies heavily on earlier work [8–14]. Specifically, the action for $\pi^+ \to e^+\nu$ [11] and $\pi^0 e^+\nu$ [12] are employed as the starting point in Section 2. Variation of this SO(3) gauge-invariant action leads to meson wave equations for the π and ρ isotriplets which give a clue to the basic decay mechanism. Some of these basic equations, notations, and variable transformations are given in the Appendix. In Section 3, the quasi-four-quark meson equations derived in ref. 14, hereafter denoted by I, for application to the decay of a vector meson into two pseudoscalar mesons $V \to PP$ are modified to apply to $P \to VV$ of interest here. The decay rate formula, dependent upon the perturbed and virtual vector meson wave functions, is given in Section 4. In Section 5, equations determining these perturbed wave functions are given. An order-of-magnitude estimate of these perturbed functions is obtained in Section 6. With this estimate, a constant associated with the strong coupling from $V \to PP$ in I, and the quark masses from meson spectra [10], the decay rate is estimated in Section 7 and found to be in order-of-magnitude agreement with data. A brief summary in relation to earlier work is given in Section 8.

2. TWO-QUARK ISOTRIPLET ACTION

The two-quark meson action (A4) has been generalized to apply to the pion triplet in Section 7 of ref. 9, where SO(3) gauge invariance has been shown. On a spherical basis, this action has been given in Section 9 of ref. 11 for $\pi^+ \to e^+\nu$ and in Section 6 of ref. 12 for $\pi^+ \to \pi^0 e^+\nu$. Putting $a = \frac{1}{2}$ there, as is done in (A5), this action reads

$$S_M = S_{F3} + S_{MT} \tag{2.1}$$

$$S_{F3} = -\tfrac{1}{4}\int d^4X \sum_1^3 G_l^{\mu\kappa}\, G_{l\mu\kappa} \tag{2.2a}$$

$$G_l^{\mu\kappa} = \partial^\mu W_l^\kappa - \partial^\kappa W_l^\mu - \varepsilon_{jkl} g W_j^\mu W_k^\kappa \tag{2.2b}$$

$$\begin{aligned} S_{MT} &= \int L_{MT}\, d^4X \\ &= -\int d^4X \int d^4x \\ &\quad \times \frac{1}{2}\left\{\begin{matrix} \left(\begin{matrix} \frac{1}{2}\{[\frac{1}{2}D^{\dot{b}a} - \partial^{\dot{b}a}]\chi^{*\dot{e}}_{Ta}\}\{[\frac{1}{2}D_{f\dot{e}} + \partial_{f\dot{e}}]\chi^f_{T\dot{b}}\} + \\ \frac{1}{2}\{[\frac{1}{2}D^{\dot{b}a} + \partial^{\dot{b}a}]\psi^{*\dot{e}}_{Ta}\}\{[\frac{1}{2}D_{f\dot{e}} - \partial_{f\dot{e}}]\psi^f_{T\dot{b}}\} + \text{h.c.} \end{matrix}\right) + \\ (\Phi_P - M_m^2)\,[\psi^{*\dot{e}}_{Ta}\chi^a_{T\dot{e}} + \text{h.c.}] \end{matrix}\right\} \end{aligned} \tag{2.3}$$

$$\chi^a_{T\dot{e}} = \begin{pmatrix} \chi_{\dot{e}}^{+a}\,(X, x) \\ \chi_{\dot{e}}^{0a}\,(X, x) \\ \chi_{\dot{e}}^{-a}\,(X, x) \end{pmatrix}, \qquad \chi \to \psi \tag{2.4a}$$

$$D^{a\dot{e}} = \partial_X^{a\dot{e}} + \frac{i}{2}\, g \begin{pmatrix} W_3 & \sqrt{2}W^+ & 0 \\ \sqrt{2}W^- & 0 & \sqrt{2}W^+ \\ 0 & \sqrt{2}W^- & -W_3 \end{pmatrix} \tag{2.4b}$$

The symbols are explained in the Appendix. g is the weak charge. The superscript * also denotes Hermitian conjugation when appropriate, as in (2.3). The superscripts +, 0, and −, hereafter denoted by t, in (2.4a) refer to the charges of the isotriplet members.

Equations (2.1)–(2.4) together with (A3) show that both weak decays mentioned above (2.1) are mediated by the charged gauge bosons $W^\pm$. The vector triplet $\rho^{\pm}$ and ρ^0, associated with $\boldsymbol{\chi}_1$ and $\boldsymbol{\psi}_1$ in (A3), decays strongly via a quasi-four-quark generalization of (A4) and has been treated in I.

An SO(3) gauge transformation has been chosen in Section 7 of ref. 9 such that the $t = +$ and $-$ components in (2.3) and (2.4a) vanish. In this case the gauge field W_3 drops out in (2.3) and can be assigned to the electromagnetic field A,

$$W_3^{a\dot{e}}(X) = A^{a\dot{e}}(X) = \delta^{a\dot{e}} A_0(X) - \vec{\sigma}^{a\dot{e}} \vec{A}(X) \tag{2.5}$$

which is well known [15]. Since the two-quark action (2.3) contains only one photon $\mathbf{A}(X)$ given in (2.5), it cannot account for $\pi^0 \to \gamma\gamma$, just as it cannot account for $\rho \to \pi\pi$, which involves four or more quarks.

However, (2.3) and (2.4) contain the prototype of the mechanism for $\pi^0 \to \gamma\gamma$. Since weak interactions are not involved here, $W^{\pm}$ in (2.2)–(2.4) can be put to zero. Further, A_0 in (2.5) can be put to zero by a suitable gauge transformation. Making use of (2.4) and (A3), (2.3) becomes

$$S_{MT} = -\frac{1}{4} \int d^4X \int d^4x$$

$$\left\{ \begin{array}{l} \left[(\chi_{\dot{e}a}^{+}*, \chi_{\dot{e}a}^{0}*, \chi_{\dot{e}a}^{-}*) \left(\frac{1}{2} \partial_X^{a\dot{b}} - i \frac{g}{4} A^{a\dot{b}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} - \partial^{a\dot{b}} \right) \right] \\ \times \left[\left(\frac{1}{2} \partial_X^{f\dot{e}} + i \frac{g}{4} A^{f\dot{e}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \partial^{f\dot{e}} \right) \begin{pmatrix} \chi_{\dot{b}f}^{+} \\ \chi_{\dot{b}f}^{0} \\ \chi_{\dot{b}f}^{-} \end{pmatrix} \right] \\ + \chi \to \psi \text{ and } \partial \to -\partial + \text{h.c.} - 2(\Phi_P - M_m^2)[\psi_T^{*a\dot{e}} \chi_{T\dot{e}a} + \text{h.c.}] \end{array} \right\} \tag{2.6}$$

Here, the ∂ operators operate on the χ's and ψ's even if they stand to their right. Restricted variation, according to Section 3 of ref. 9 of (2.6) with respect to χ_0^{0*} and ψ_0^{0*}; together with (A3), gives back (A1) with $\boldsymbol{\chi}_1$ and $\boldsymbol{\psi}_1$ put to zero,

$$\left(\frac{1}{2} \partial_{X\dot{b}a} - \partial_{\dot{b}a} \right) \left(\frac{1}{2} \partial_{X\dot{e}f} + \partial_{\dot{e}f} \right) \delta^{\dot{b}f} \chi_0^0 + (\phi_P - M_m^2) \delta_{\dot{e}a} \psi_0^0 = 0$$

$$\left(\frac{1}{2} \partial_{X\dot{c}b} + \partial_{\dot{c}b} \right) \left(\frac{1}{2} \partial_{X\dot{e}d} - \partial_{\dot{e}d} \right) \delta^{d\dot{c}} \psi_0^0 + (\phi_P - M_m^2) \delta_{\dot{e}b} \chi_0^0 = 0 \tag{2.7}$$

for π^0. Restricted variation of (2.6) with respect to $\boldsymbol{\chi}_1^{\pm *}$ and $\boldsymbol{\psi}_1^{\pm *}$ yields (A1) with $\chi_0^{\pm}$ and $\psi_0^{\pm}$ put to zero and with the $\mathbf{A}$ terms included,

$$\left(\frac{1}{2}\partial_{X\dot{b}a} - \partial_{\dot{b}a}\right)\left(\frac{1}{2}\partial_{X\dot{e}f} + \partial_{\dot{e}f}\right)\vec{\sigma}^{\dot{b}f}\vec{\chi}_1^{\pm} - (\phi_P - M_m^2)\vec{\sigma}_{\dot{e}a}\vec{\psi}_1^{\pm}$$
$$= \pm i\frac{g}{8}\left(\vec{\sigma}_{\dot{b}a}\vec{A}\left(\frac{1}{2}\partial_{X\dot{e}f} + \partial_{\dot{e}f}\right) + \left(\frac{1}{2}\partial_{X\dot{b}a} - \partial_{\dot{b}a}\right)\vec{\sigma}_{\dot{e}f}\vec{A}\right.$$
$$\left. + \left(\frac{1}{2}\partial_{X\dot{e}f} + \partial_{\dot{e}f}\right)\vec{\sigma}_{\dot{b}a}\vec{A} + \vec{\sigma}_{\dot{e}f}\vec{A}\left(\frac{1}{2}\partial_{X\dot{b}a} - \partial_{\dot{b}a}\right)\right)\delta^{f\dot{b}}\chi_0^{\pm} \tag{2.8a}$$

$$\left(\frac{1}{2}\partial_{X\dot{c}b} + \partial_{\dot{c}b}\right)\left(\frac{1}{2}\partial_{X\dot{e}d} - \partial_{\dot{e}d}\right)\vec{\sigma}^{d\dot{c}}\vec{\psi}_1^{\pm} - (\Phi_P - M_m^2)\vec{\sigma}_{\dot{e}b}\vec{\chi}_1^{\pm}$$
$$= \pm i\frac{g}{8}\left(\vec{\sigma}_{\dot{c}b}\vec{A}\left(\frac{1}{2}\partial_{X\dot{e}d} - \partial_{\dot{e}d}\right) + \left(\frac{1}{2}\partial_{X\dot{c}b} + \partial_{\dot{c}b}\right)\vec{\sigma}_{\dot{e}d}\vec{A}\right.$$
$$\left. + \left(\frac{1}{2}\partial_{X\dot{e}d} - \partial_{\dot{e}d}\right)\vec{\sigma}_{\dot{c}b}\vec{A} + \vec{\sigma}_{\dot{e}d}\vec{A}\left(\frac{1}{2}\partial_{X\dot{c}b} + \partial_{\dot{c}b}\right)\right)\delta^{d\dot{c}}\psi_0^{\pm} \tag{2.8b}$$

Here, terms of order g^2 and higher have been dropped.

The π^0 described by (2.7) can only decay via the strong interaction term Φ_p, whose source is a product of two meson wave functions according to (A2). The triplet components of these meson wave functions are assigned to $\boldsymbol{\chi}_1^{\pm}$ and $\boldsymbol{\psi}_1^{\pm}$ in (2.8), which represent virtual intermediate vector mesons $\rho^{\pm}$. These have as their source the photon field **A** coupled to the singlet components $\chi_0^{\pm}$ and $\psi_0^{\pm}$, which are assigned to virtual final-state pseudoscalar mesons $\pi^{\pm}$ to be annihilated. This provides the prototype of the mechanism, which is $\pi^0(\mathrm{P}) \to \rho^+\rho^-(\mathrm{VV}) \to \gamma\gamma$ plus π^+ annihilating π^-.

3. QUASI- FOUR-QUARK WAVE EQUATIONS FOR MESONS

Let the two quarks in the intial π^0 be u and $\bar{\mathrm{u}}$ at first. This state is perturbed by the creation of a pair of virtual d and $\bar{\mathrm{d}}$ quarks as illustrated in Fig. 1a. The result of the following treatment from here to (4.4) is complemented by an equivalent treatment with u $\leftrightarrow$ d to represent the treatment of π^0.

The strong interaction perturbation in Figs. 1a and 2a is analogous to that of Fig. 1a of I for a vector meson decaying into two pseudoscalar mesons V $\to$ PP according to the OZI rule. This process is similar to the P $\to$ VV process mentioned at the end of Section 2 and depicted in Figs. 1a, 1b, 2a, and 2b. Therefore, the mathematical developments in Section 4–7 of I can with the interchange V $\leftrightarrow$ P be appropriately taken over here. Note that all these figures are quasiclassical illustrations of the mathematical representations and are not to be taken too literally.

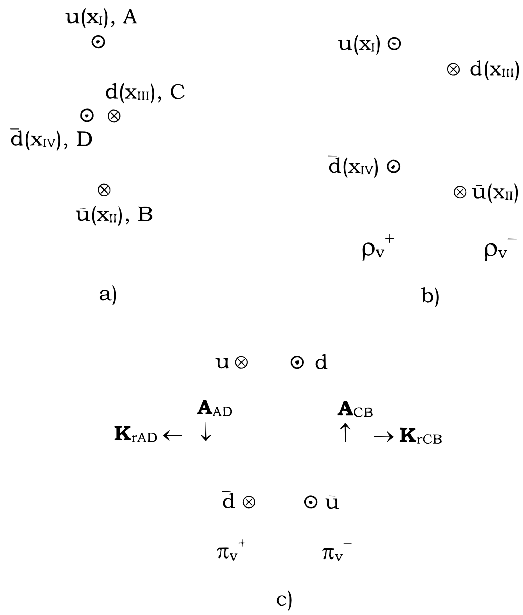

Fig. 1. Quasiclassical illustration of $\bar{u}u \to 2\gamma$. The same set of figures with u ↔ d illustrates $\bar{d}d \to 2\gamma$. Combining both sets shows $\pi^0 \to 2\gamma$. Here ⊙ denotes spin up and ⊗ denotes spin down. In the initial stage (a), a pair of virtual $\bar{d}$ and d quarks are created at the origin. In the intermediate stage (b), $\bar{d}$ (d) has moved closer to the $\bar{u}$ (u) quark so that the assumption $x_I \approx x_{III}$ and $x_{II} \approx x_{IV}$ implied in (3.1) and which underlies the quasi-four-quark meson equations in Section 3 holds approximately. These quarks represent two virtual charged vector mesons ρ_v^+ and ρ_v^-. In the final stage (c), the $\bar{u}$ and u quarks flip their spins so that the both ρ_v turn into a pair of virtual pions π_v^+ and π_v^-, which in turn annihilate each other. The spin and energy released turn into two photons $\mathbf{A}_{AD}$ and $\mathbf{A}_{BC}$ with momenta $\mathbf{K}_{AD}$ and$\mathbf{K}_{BC}$, respectively.

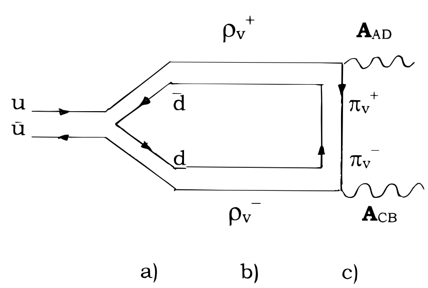

Fig. 2. Representation of Fig. 1 for $\pi^0 \to 2\gamma$ in a quark line diagram. (a), (b), and (c) in both figures represent the same initial, intermediate, and final stages, respectively.

As in Section 4 of I, let the u- and d-quark wave functions be $\chi_{A\dot{b}}(x_{\rm I})\xi^1_A(z_{\rm I})$ and $\chi_{C\dot{b}}(x_{\rm III})\xi^2_C(z_{\rm III})$, respectively (see Fig. 1a for notation). In the intermediate stage of Fig. 1b, $x_{\rm III} \approx x_{\rm I}$ and $z_{\rm III} \approx z_{\rm I}$, so that the zeroth-order u-quark wave function can be approximately considered to be mixed with a first-order perturbational d-quark amplitude, as in (I 4.1):

$$\chi_{A\dot{b}}(x_{\rm I})\xi^1_A(z_{\rm I}) + \chi_{C\dot{b}}(x_{\rm I})\,\xi^2_C(z_{\rm I}), \qquad \psi^b_A(x_{\rm I})\xi^1_A(z_{\rm I}) + \psi^b_C(x_{\rm I})\xi^2_C(z_{\rm I}) \tag{3.1}$$

A similar set holds for the zeroth-order antiquark $\bar{\rm u}$, denoted by B, and first-order antiquark $\bar{\rm d}$, denoted by D. The approximation (3.1) together with its antiquark version for B and D allows for an approximate treatment of the four-quark problem in terms of the more tractable two-quark one.

The development of Section 4 of I leads to (I 4.6a) and (I 4.7)–(I 4.9), or

$$\partial^{a\dot{b}}_{\rm I}\;\partial_{{\rm II}\dot{e}f}\chi^{f}_{AB\dot{b}} = (\phi_{PAB} + \phi_{1P4} - M^2_{AB})\psi^a_{AB\dot{e}} \tag{3.2a}$$

$$\partial_{{\rm I}\dot{c}b}\;\partial^{d\dot{e}}_{\rm II}\;\psi^b_{AB\dot{e}} = (\phi_{PAB} + \phi_{1P4} - M^2_{AB})\chi^d_{AB\dot{c}} \tag{3.2b}$$

$$(3.2) \text{ with } AB \to AD \tag{3.3}$$

$$(3.2) \text{ with } AB \to CB \tag{3.4}$$

Here, (3.2) is of zeroth order and (3.3) and (3.4) are of first order. ϕ_{PAB} is the zeroth order interquark potential, which is

$$\Phi_{PAB} = -d_m/r - \Phi_0 \tag{3.5}$$

for a free meson according to Section 2a of I. The perturbed potential Φ_{1P4} is given by (I 4.6c),

$$\Box_{\rm I}\Box_{\rm II}\Phi_{1P4} = \frac{1}{16}\,g^4_q\,[\psi^a_{AD\dot{b}}\chi^{\dot{b}}_{CBa} + \psi^a_{CB\dot{b}}\chi^{\dot{b}}_{ADa} + \ldots] + \text{c.c.} \tag{3.6}$$

where . . . denotes terms that subsequently drop out in Section 6 of I. g_q of (A2) has been reinstated here. Φ_{1P4} is of second order according to (3.3), (3.4), and (3.6).

For V $\to$ PP, (3.2) has been converted into the action (I 5.1), which, apart from Φ_{1P4}, is the same as the $t = 0$ part of (2.6). The development in the rest of Section 5 of I can be taken over here if the vector meson (V) wave functions there are replaced by the pseudoscalar meson (P) wave functions here. The decay amplitude (I 5.6) is modified to

$$S_{fi} = -i\,\frac{4}{E_{00}\Omega_N} \times \int d^4X\,d^3\vec{x}\;\Phi_{1P4}(\exp(iE_{00}X^0))|\psi_{00}(\vec{x})|^2 \Big/ \int d^3\vec{x}\;|\psi_{00}(\vec{x})|^2 \tag{3.7}$$

where Ω_N is the large normalization volume of π^0, E_{00} the π^0 mass, and ψ_{00} the wave function (I 2.2a) of π^0. Further, $a_{AB}^{(1)}$ and $|V_{AB}\rangle$ in Section 5 of I have been replaced by $a_{AB}^{(2)}$ and $|\pi^0\rangle$, respectively.

4. PERTURBED POTENTIAL AND DECAY RATE FORMULA

By (3.2) and (3.5), the perturbed potential in (3.6) must satisfy (I 6.1) modified to

$$|\Phi_{1P4}| << |\Phi_{PAB}| \approx |d_m/\langle r_0\rangle - \Phi_0| \cong (0.75 + 0.24) \text{ GeV} \tag{4.1}$$

where (I 2.5a) and (I 2.6a) have been consulted. The physical picture of Section 6a of I is replaced by Figs. 1 and 2 here. The development of (I 6.3) to (I 7.4) holds here if the switch of V $\rightarrow$ PP to P $\rightarrow$ VV indicated below Fig. 2 is carried out. Specifically, the subscript 0 referring to the real final state pseudoscalar mesons in I is replaced by 1, which refers to the virtual and intermediate vector mesons of Figs. 1b and 1c. As in (3.7), the initial vector meson mass E_{10} is replaced by E_{00}. Equations (I 6.5) and (I 6.9) now become

$$\Phi_{1P4}(x_{\rm I}, x_{\rm II}) = \Phi_{1P}(\vec{x}) \exp[i(\vec{K}_{1AD} - \vec{K}_{1CB})\vec{X} - i(E_{1\vec{K}AD} - E_{1\vec{K}CB})X^0] \tag{4.2a}$$

$$\begin{aligned} I_{1P}(\vec{x}) &= 8\pi E_{00}\, \Phi_{1P}(\vec{x}) \\ &= -\frac{1}{4} g_q^4 \int d^3\vec{x}\,' \, [\vec{\psi}_{1\vec{K}AD}(\vec{x}\,')\vec{\chi}^*_{1\vec{K}CB}(\vec{x}\,') \\ &\quad + \vec{\chi}^*_{1\vec{K}AD}(\vec{x}\,')\vec{\psi}_{1\vec{K}CB}(\vec{x}\,')] \sin\left(\frac{1}{2} E_{00}|\vec{x} - \vec{x}\,'|\right) \end{aligned} \tag{4.2b}$$

Here, use has been made of

$$\begin{aligned} \psi_{AD}^{a\dot{e}} &= \delta^{a\dot{e}}\psi_{0AD} - \vec{\sigma}^{a\dot{e}}\vec{\psi}_{1AD} \\ \psi_{0AD} &= \psi_{00AD}(\vec{x}) \exp(-iE_{0AD}X^0) \\ \vec{\psi}_{1AD} &= \vec{\psi}_{1\vec{K}AD}(\vec{x}) \exp(-iE_{1\vec{K}AD}X^0 + i\vec{K}_{1\vec{K}AD}\vec{X}) \\ \psi &\rightarrow \chi, \qquad AD \rightarrow CB \end{aligned} \tag{4.3}$$

which refers to virtual mesons and is hence more general than (3.1) of ref. 13 used in (I 6.3), which is associated with real mesons. In the bracket of (4.2b), the singlet product terms have been dropped because only the triplet terms will couple to the final state photons **A**, as is indicated in (2.8).

In the special case of $\boldsymbol{\psi}_1(\mathbf{x}) = \boldsymbol{\psi}_1(r = |\mathbf{x}|)$ together with $\boldsymbol{\psi}_1 \rightarrow \boldsymbol{\chi}_1$, the angular integrations in (4.2b) can be carried out as in (I 6.10) to yield

$$\Phi_{1P}(r) = -\frac{g_q^4}{E_{00}^4}\left\{\begin{array}{l}\dfrac{\cos R}{R}\displaystyle\int_0^R dR'\, R'(\sin R' - R'\cos R')V(r') \\ \quad + \sin R \displaystyle\int_0^R dR'\, R'V(r')\sin R' \\ \quad - \cos R \displaystyle\int_R^\infty dR'\, R'V(r')\cos R' \\ \quad + \dfrac{\sin R}{R}\displaystyle\int_R^\infty dR'\, R'(\cos R' + R'\sin R')V(r')\end{array}\right\} \tag{4.4a}$$

$$V(r) = \vec{\psi}_{1\vec{K}AD}(r)\vec{\chi}^*_{1\vec{K}CB}(r) + \vec{\chi}^*_{1\vec{K}AD}(r)\vec{\psi}_{1\vec{K}CB}(r), \qquad R = E_{00}r/2 \tag{4.4b}$$

Inserting (4.2) into (3.7) leads to the equivalent of (I 7.1)

$$S_{fi} = \frac{i8\pi}{E_{00}\Omega_N}\int d^3\vec{X}\,(\exp i(\vec{K}_{1AD} - \vec{K}_{1CB})\vec{X})\delta\,(E_{00} - E_{1\vec{K}AD} + E_{1\vec{K}CB})\overline{\Phi}_{1P} \tag{4.5a}$$

$$\overline{\Phi}_{1P} = g_q^4\int d^3\vec{x}\,\psi_{00}^2(r)\Phi_{1P}(\vec{x})\Big/\int d^3\vec{x}\,\psi_{00}^2(r) \tag{4.5b}$$

The wave function of a free pseudoscalar meson at rest, such as π^0 here, is given by (4.3), (I 2.2a), (2.3a), (2.5a), and (B7) of ref. 13. We have

$$\psi_{AB}^{a\dot{e}} = \delta^{a\dot{e}}\psi_{0AB}, \qquad \psi \to \chi$$

$$\psi_{0AB} = -\chi_{0AB} = \psi_{00}(r)\exp(-iE_{00}X^0)$$

$$\psi_{00}(r) = \sqrt{d_m^3/8\pi\Omega_N}\,\exp(-d_m r/2), \qquad d_m = 0.864\text{ GeV} \tag{4.6}$$

The decay rate (I 7.2) here goes over to

$$\Gamma(\pi^0 \to \gamma\gamma) = \sum_{\text{final states}}\frac{|S_{fi}|^2}{T_d} = \frac{\Omega_N^2}{64\pi^6}\int d^3\vec{K}_{1AD}\int d^3\vec{K}_{1CB}\,\frac{|S_{fi}|^2}{T_d} \tag{4.7}$$

5. EQUATIONS FOR THE PERTURBED WAVE FUNCTIONS

To obtain the decay rate, $\Phi_{1P}(\mathbf{x})$ in (4.5) and $\boldsymbol{\psi}_1$, and $\boldsymbol{\chi}_1$, in (4.2b) or (4.4b) need be evaluated. From here on, the development goes beyond that of V $\to$ PP of I with V $\leftrightarrow$ P, and is associated with Figs. 1c and 2c.

Keeping only the singlet part of ψ and χ in (3.2), i.e., dropping $\boldsymbol{\psi}_1$ and $\boldsymbol{\chi}_1$ in (A3), we have that (3.2) represents a generalization of the two-quark meson equation (2.7) to the quasi-four-quark case. The difference is simply the introduction of the second-order perturbational potential Φ_{1P4}. The generalization of the two-quark equations (2.8) for virtual and charged vector mesons to quasi-four-quark equations is here identified as (3.3) and (3.4) generalized to include electromagnetic gauge fields. Figures 1b and 2b show that the virtual vector meson ρ_v^+ is associated with *AD* and ρ_v^- with *CB*, observing the complementary interchange u ↔ d there. These correspond to $\boldsymbol{\chi}_1^{\pm}$ and $\boldsymbol{\psi}_1^{\pm}$ in (2.8), which are of first order in the weak charge *g*. Introduce the electromagnetic gauge field *A* via

$$\partial_X^{a\dot{b}} \to \partial_X^{a\dot{b}} + ieA_{AD}^{a\dot{b}}(X) \to \partial_X^{a\dot{b}} - ie\vec{\sigma}^{a\dot{b}}\vec{A}_{AD}(X), \qquad AD \to CB \quad (5.1)$$

into (3.3) and (3.4). Here, *e* is the positron charge, (2.5) and (A3) have been consulted, and the time component A_0 has been dropped, as was done above (2.6). Equations (3.3) and (3.4) now become

$$\left(\frac{1}{2}\partial_{X\dot{b}a} - \partial_{\dot{b}a}\right)\left(\frac{1}{2}\partial_{X\dot{e}f} + \partial_{\dot{e}f}\right)\vec{\sigma}^{\dot{b}f}\vec{\chi}_{1AD} - (\Phi_{PAB} + \Phi_{1P4} - M_{AD}^2)\,\vec{\sigma}_{\dot{e}a}\,\vec{\psi}_{1AD}$$

$$= i\frac{e}{2}\left(\vec{\sigma}_{\dot{b}a}\vec{A}_{AD}\left(\frac{1}{2}\partial_{X\dot{e}f} + \partial_{\dot{e}f}\right) + \left(\frac{1}{2}\partial_{X\dot{b}a} - \partial_{\dot{b}a}\right)\vec{\sigma}_{\dot{e}f}\vec{A}_{AD}\right)\delta^{f\dot{b}}\,\chi_{0AD} \quad (5.2a)$$

$$\left(\frac{1}{2}\partial_{X\dot{c}b} + \partial_{\dot{c}b}\right)\left(\frac{1}{2}\partial_{X\dot{e}d} - \partial_{\dot{e}d}\right)\vec{\sigma}^{d\dot{c}}\,\vec{\psi}_{1AD} - (\Phi_{PAB} + \Phi_{1P4} - M_{AD}^2)\,\vec{\sigma}_{\dot{e}b}\,\vec{\chi}_{1AD}$$

$$= i\frac{e}{2}\left(\vec{\sigma}_{\dot{c}b}\vec{A}_{AD}\left(\frac{1}{2}\partial_{X\dot{e}d} - \partial_{\dot{e}d}\right) + \left(\frac{1}{2}\partial_{X\dot{c}b} + \partial_{\dot{c}b}\right)\vec{\sigma}_{\dot{e}d}\vec{A}_{AD}\right)\delta^{f\dot{b}}\,\psi_{0AD} \quad (5.2b)$$

$$\text{(5.2) with } AD \to CB \text{ and } e \to -e \quad (5.3)$$

Since the u- and d-quark masses are nearly the same, as is shown in Table 1 of ref. 10, M_{AB}, M_{AD}, and M_{CB} can all be considered as equal to M_m. Note that $e = g/2$ in the limit of SU(3) flavor symmetry according to (5.2) of ref. 12. Therefore, (5.2) and (5.3) here play the role of (2.8); the difference resides largely in the presence of the second-order perturbational potential Φ_{1P4}, which may be neglected here.

As in (3.3) of ref. 13, let

$$\vec{A}_{AD}(X) = (2E_{rAD}\,\Omega)^{-1/2}\sum_T \vec{e}_{TAD}\exp(-iE_{rAD}X^0 + i\vec{K}_{rAD}\vec{X}), \qquad AD \to CB \quad (5.4)$$

where the subscript *r* refers to **A**. Here Ω is a large normalization volume for **A** and $\mathbf{e}_T$ its polarization vector with $T = 1, 2$. Inserting (5.4), (4.3), and (3.5) into (5.2) and (5.3) and making use of (A5) leads to

$$\left(\frac{1}{4} E^2_{1\vec{K}AD} - \frac{1}{4} \vec{K}^2_{1AD} - \nabla^2\right)\vec{\chi}_{1\vec{K}AD} + \frac{1}{2} \vec{K}_{1AD}(\vec{K}_{1AD}\vec{\chi}_{1\vec{K}AD})$$

$$+ 2\vec{\partial}(\vec{\partial}\vec{\chi}_{1\vec{K}AD}) + E_{1\vec{K}AD}\vec{\partial} \times \vec{\chi}_{1\vec{K}AD} + (d_m/r - \Phi_0 - M_m^2)\vec{\psi}_{1\vec{K}AD}$$

$$= \frac{e}{4\sqrt{2E_{rAD}\Omega}} [(2E_{0AD} + E_{rAD})\, \vec{e}_{TAD} + 4\vec{\partial} \times \vec{e}_{TAD}$$

$$- i\vec{K}_{rAD} \times \vec{e}_{TAD}]\chi_{00AD} \tag{5.5a}$$

(5.5a) with $\chi \leftrightarrow \psi$ and $\vec{\partial}\times \to -\vec{\partial}\times$ (5.5b)

(5.5a) with $AD \leftrightarrow CB$ and $g \to -g$ (5.6a)

(5.6a) with $\chi \leftrightarrow \psi$ and $\vec{\partial}\times \to -\vec{\partial}\times$ (5.6b)

$$E_{1\vec{K}AD} = E_{0AD} + E_{rAD} \qquad \vec{K}_{1AD} = \vec{K}_{rAD} \qquad AD \to CB \tag{5.7}$$

In (2.7) and (2.8), ψ_0^0, ψ_0^+, and ψ_0^- refer to the same pseudoscalar meson wave function due to the invariance of (2.3) under SO(3) gauge transformations. They therefore have the same functional dependence, but may be different in their amplitudes. In the beginning of this section, it was indicated that that π^0 wave function ψ_{0AB} in (3.2) is the same as ψ_0^0 in (2.7). Above (5.4), it was pointed out that (5.2) and (5.3) take the place of (2.8). Therefore, ψ_{0AD}, ψ_{0CB}, ψ_{0AB}, and ψ_0^0 all should have the same functional dependence, but can differ in their amplitudes. By (4.6) one has

$$\psi_{0AD} = -\chi_{0AD} = [\sqrt{d_m^3/8\pi\Omega_{AD}}\, \exp(-d_m r/2)]\exp(-iE_{00}X^0),$$
$$AD \to CB \tag{5.8a}$$

$$E_{0AB} = E_{0AD} = E_{0CB} = E_{00} \tag{5.8b}$$

While $\Omega_N \to \infty$ for the free π^0 at rest in (4.6) according to Section 2a of I, Fig. 1c shows that the virtual π_v^+ and π_v^-, represented by ψ_{0AD} and ψ_{0CB}, respectively, are close to each other and hence not free. According to Section 2b of I, the confining potential Φ'_{P00} in (I 2.1b,c) no longer vanishes and $\Omega_N \to$ some finite volume Ω_c according to (I 3.7a) ff. Therefore, Ω_{AD} and Ω_{CB} of the virtual final state $\pi_v^\pm$ are finite. This volume is of the order of the π^0 volume in the relative space according to Fig. 1c, which volume is also the only volume scale available. Using (4.6), it becomes

$$\Omega_{AD} = \Omega_{CB} = \int d^3\vec{x}\, \exp(-d_m r) = 8\pi/d_m^3 \tag{5.9}$$

Analogous to (I 6.3), negative-energy solutions will be chosen for $\mathbf{A}_{CB}$ so that this photon will propagate in the direction opposite to that of $\mathbf{A}_{AD}$,

$$E_{rCB} = -|E_{rCB}| = -|\vec{K}_{rCB}| \tag{5.10}$$

This will make E_{rAD} and E_{rCB} in (5.5) and (5.6), respectively, to appear with different signs. Thus, the magnitude of $\boldsymbol{\chi}_{1AD}$ for ρ_v^+ will differ from that of $\boldsymbol{\chi}_{1CB}$ for ρ_v^-. This asymmetry is compensated for when u ↔ d mentioned above Fig. 1 is carried out. Furthermore, $\rho_v^\pm$ are virtual, so that this asymmetry is not observable. Finally, (4.5), (5.7), and (5.10) show that

$$\vec{K}_{1AD} = \vec{K}_{1CB} = \vec{K}_{rAD} = \vec{K}_{rCB} = \vec{K}_{\vec{0}}, \qquad K_0 = E_{rAD} = |E_{rCB}| \tag{5.11}$$

6. ESTIMATE OF PERTURBED WAVE FUNCTIONS

The right sides of (5.5) and (5.6) are now known and the virtual vector meson wave functions $\boldsymbol{\chi}_{1AD}$, etc., are in principle fixed. With (4.2), (4.3), and (4.5)–(4-7), the decay rate is determined. However, (5.5) and (5.6) are too complex to solve. The are of the same type of inhomogeneous equations as (B8b) and (B8c) of ref. 13, which were also not amenable to analytic solution. Instead, the order of magnitude of the unknown wave functions was estimated by means of a dimensional analysis approximation.

This approximation is somewhat modified for application to (5.5) and (5.6). Figures 1b and 1c indicate that **A** is parallel to the vector **x** connecting the quark and the antiquark in each of the virtual vector mesons. Therefore,

$$\hat{r} = \vec{x}/|\vec{x}| \qquad \text{parallel to } \vec{e}_T \text{ or } \vec{A}(X) \tag{6.1}$$

will be assumed. Mathematically, (6.1) is inconsistent because **x** is the relative space coordinate of the quarks which is independent of the laboratory coordinate X upon which **A** depends. An extenuating circumstance is that although $\mathbf{e}_r$ is a vector in **X** space, it is a constant directional vector independent of X. This assumption allows for a simple order-of-magnitude estimate of the virtual vector meson wave functions in (5.5) and (5.6).

From (6.1), (4.3), and (5.8a), it is now seen that the middle term on the right of each member of (5.5) and (5.6) drops out. It can also be seen from (5.7), (5.8), (5.10), (5.11), and the δ function in (4.5a) that

$$K_0 = E_{0AD}/2 \tag{6.2}$$

Therefore, the ratio of the last and imaginary term to the first term on the right of (5.5) and (5.6) is 1/5 and 1/3, respectively. Further, these terms are perpendicular to each other. Therefore, the imaginary terms in (5.5) and (5.6) may be neglected at first and reintroduced in (6.4) below as correction. Noting that $\mathbf{K}_r \perp \mathbf{e}_T$, it turns out that

$$\vec{\chi}_{1\vec{K}AD} = -\vec{\psi}_{1\vec{K}AD} = \hat{r}B_{AD}\exp(-d_m \mathrm{r}/2), \qquad AD \to CB \tag{6.3}$$

are solutions to (5.5) and (5.6). These equations together with (4.3), (4.6), and (5.7)–(6.2) lead to

$$\hat{r}B_{AD} = \frac{d_m^3}{8\pi}\frac{1}{(E_{00}^2/4 + L_0)}\frac{e}{4\sqrt{E_{00}\Omega}}\left(\frac{5}{2}E_{00}\vec{e}_T - i\vec{K}_{rAD}\times\vec{e}_T\right) \quad (6.4a)$$

$$\hat{r}B_{CB} = -\frac{d_m^3}{8\pi}\frac{1}{L_0}\frac{e}{4\sqrt{E_{00}\Omega}}\left(\frac{3}{2}E_{00}\,\vec{e}_T - i\vec{K}_{rCB}\times\vec{e}_T\right) \quad (6.4b)$$

$$L_0 = d_m^2/4 + \Phi_0 + M_m^2 \quad (6.4c)$$

7. DECAY RATE

Since (6.3) depends only upon r, (4.4b) together with (5.10), (6.3), and (6.4) becomes

$$V(r) = \frac{e^2E_{00}}{\Omega}\frac{d_m^6}{128\pi^2}\frac{1}{L_0(E_{00}^2/4 + L_0)}\exp\left(-d_m r\right) \quad (7.1)$$

Inserting (7.1) into (4.4a) and carrying out the integrals leads to

$$\begin{aligned} I_{1P}(r) &= 8\pi E_{00}\Phi_{1P} \\ &= \frac{1}{16\pi}\frac{e^2d_m^6}{\Omega\,E_{00}^2}\frac{1}{L_0(E_{00}^2/4 + L_0)}\frac{1}{1 + c^2} \\ &\times\left[-\left(2 + \frac{8c}{(1 + c^2)R}\right)\exp(-cR) + \operatorname{Re}\frac{8c}{(1 + c^2)R}\exp(iR)\right. \quad (7.2) \\ &\left. + \operatorname{Im} 2c\exp(iR)\right], \qquad c = \frac{2d_m}{E_{00}} \end{aligned}$$

Inserting (7.2) into (4.5b) yields

$$\bar{I}_{1P}(r) = 8\pi E_{00}\overline{\Phi}_{1P} = \frac{e^2}{\Omega}f_c \quad (7.3a)$$

$$\begin{aligned} f_c &= \frac{1}{256\pi}\frac{d_m^3E_{00}}{L_0(E_{00}^2/4 + L_0)}\frac{c^6}{(1 + c^2)^2} \\ &\times\left[\frac{4c(5c^2 - 3)}{(1 + c^2)^3} - \frac{2}{c(1 + c^2)^2} - \frac{1}{2c^3}\right] \quad (7.3b) \end{aligned}$$

The integral in (4.5a) is carried out to obtain S_{fi}, which is then inserted into (4.7). After carrying out the integrations with the aid of (I 7.3), (I 7.7), and the last of (I 7.8), (4.7) becomes

$$\Gamma(\pi^0 \to \gamma\gamma) = C_I e^4 f_c^2 / 8E_{00}^2 \tag{7.4a}$$

$$C_I = g_q^8 M_{ds}^3 / 4\pi^3, \qquad M_{ds}^3 = \Omega_d / \Omega_N^2 = \text{finite} \tag{7.4b}$$

$$\Omega_d = \int d^3 \vec{X} \, (\exp i(\vec{K}_{1AD} - \vec{K}_{1CB})\vec{X}) = 8\pi^3 \, \delta(\vec{K}_{1AD} - \vec{K}_{1CB})$$

$$= (2\pi\delta(0))^3 \tag{7.4c}$$

The π^0 mass E_{00} is 0.0135 Gev, e^2 is $4\pi/137$, and d_m has been given in (4.6). From the pseudoscalar meson masses, $\Phi_0 = 0.24455$ GeV according to (I 2.5b) and M_m, the average of u- and d-quark masses, is 0.6963 GeV obtained in Table 1 of ref. 10. The C_1 in (7.4b) is associated with the strong interaction vertex P $\to$ VV indicated in Fig. 2a and is of the same nature as that for V $\to$ PP in I. Comparison of the predicted and measured $\Gamma(\varphi \to K^+ K)$ in Section 8a of I gives $C_1 = 0.35$ GeV^3. With these numbers, (7.4a,b), (7.3b), (7.2), and (6.4c) yield

$$\Gamma(\pi^0 \to \gamma\gamma) = 19.2 \text{ eV} \qquad (\text{data: } 7.74 \text{ eV}) \tag{7.5}$$

There are three error sources. In the first place, C_1 can be off by 10–20% according to Section 8a of I. This is due to the inability to calculate the Δ_2 term in (I 7.5a). This inability is of the same type of difficulty as in solving (5.5) and (5.6) here. This leads to the second and possibly the major error source, which arises from the approximation resulting from the assumption (6.1). This assumption tends to an overestimate of (6.3) since admixture of the opposite of (6.1), i.e., $\hat{r} \perp \mathbf{A}$, cannot be excluded for all $\mathbf{x}$ values. Such an admixture will, according to (6.4), lower (6.3) and hence also the decay rate (7.4a). This overestimating tendency is in agreement with (7.5). Lastly, some correction to the volume (5.9) cannot be excluded off-hand. In spite of these error sources, (7.5) is in order-of-magnitude agreement with data.

8. SUMMARY

The decay rate (4.7) together with (4.5) and (4.6) is exact, to the degree that (3.1) holds, and has been derived from a first principles, i.e., Lorentz- and gauge-invariant theory [8, 9]. This theory is nonlocal and has also accounted for a number of basic low-energy mesonic phenomena mentioned under (i)–(iv) in Section 1. Therefore, the present treatment differs basically from all earlier ones, which are based upon phenomenological and local models including form factors. These models cannot account for the phenomena (i)–(iv) mentioned above.

The only free parameter is C_I in (7.4b), which is associated with the P $\to$ VV strong coupling and the ratio of two large-volume quantities. Since there is no basic difference between P and V in the spinor strong interaction theory, C_1 has been fixed from V $\to$ PP in Section 8 of I. That the decay rate (7.5) is only an order-of-magnitude estimate is due largely to the fact that (5.5) and (5.6) can presently not be solved for insertion into (4.2b) or (4.4b). This is mainly a mathematical problem, not any difficulty in principle.

The decay mechanism is $\pi^0 \to \rho^+\rho^- \to \gamma\gamma$(plus π^+ and π^- which annihilate each other), as is illustrated in Figs. 1 and 2. This is so because the Lagrangian in (2.6) is a scalar of quadratic form. Therefore, the vector representing the photon **A** must couple to another vector. The only other vector available is that representing the virtual vector meson. In this way, the charged virtual vector mesons naturally take on the role of intermediate states and need not be assumed. The decay mechanism in the present theory agrees in principle with that of the vector meson dominance (VMD) models [2–5], but only ρ^+ and ρ^- can participate. It may be considered as a variation of the first model [1] if the intermediate proton–antiproton are replaced by ρ^+ and ρ^-. The present mechanism can, however, find no relation to the current algebra or quark models of type in refs. 6 and 7.

Finally, form factors introduced in the phenomenological models are inherent in the present nonlocal theory and contained in the integrals over the relative space **x**.

APPENDIX. TWO-QUARK MESON EQUATIONS AND ACTION

In the spinor strong interaction theory, two-quark mesons are described by the meson equations (4.11) and (4.12) of ref. 8,

$$\partial_{\mathrm{I}}^{a\dot{b}}\partial_{\mathrm{II}\dot{e}f}\,\chi_{\dot{b}}^{f}\,(x_{\mathrm{I}}, x_{\mathrm{II}}) = (\Phi_P(x_{\mathrm{I}}, x_{\mathrm{II}}) - M_m^2)\psi_{\dot{e}}^{a}(x_{\mathrm{I}}, x_{\mathrm{II}})$$

$$\partial_{I\dot{c}b}\partial_{\mathrm{II}}^{d\dot{e}}\,\psi_{\dot{e}}^{b}\,(x_{\mathrm{I}}, x_{\mathrm{II}}) = (\Phi_P(x_{\mathrm{I}}, x_{\mathrm{II}}) - M_m^2)\chi_{\dot{c}}^{d}(x_{\mathrm{I}}, x_{\mathrm{II}}) \qquad \text{(A1)}$$

$$\Box_{\mathrm{I}}\Box_{\mathrm{II}}\Phi_P(x_{\mathrm{I}}, x_{\mathrm{II}}) = \tfrac{1}{2}g_q^4\,\mathrm{Re}(\psi_{\dot{b}}^{a}(x_{\mathrm{II}}, x_{\mathrm{I}})\chi_{a}^{*\dot{b}}\,(x_{\mathrm{II}}, x_{\mathrm{I}})) \qquad \text{(A2)}$$

where Φ_P is the interquark potential, $g_q^2 = g_a g_b$ is the strong interquark coupling, M_m is the average quark mass, and x_{I} and x_{II} are the quark coordinates. χ and ψ are the meson wave functions and can be written out in the form

$$\chi^{a\dot{e}} = \delta^{a\dot{e}}\chi_0 - \vec{\sigma}^{a\dot{e}}\vec{\chi}_1, \qquad \chi \to \psi \qquad \text{(A3)}$$

At rest, the time or singlet components χ_0 and ψ_0 represent a pseudoscalar meson 0^- and the space or triplet components $\boldsymbol{\chi}_1$ and $\boldsymbol{\psi}_1$ a vector meson 1^-. In motion, the pair χ_0, ψ_0 and the pair $\boldsymbol{\chi}_1$, $\boldsymbol{\psi}_1$ are the "large" and "small" components of 0^- and "small" and "large" components for 1^-, respectively. The spinor indices in an equation can be raised or lowered if one notes that

$$\alpha^a \beta_a = -\alpha_a \beta^a, \qquad \alpha^{\dot a} \beta_{\dot a} = -\alpha_{\dot a} \beta^{\dot a}$$

Equation (A1) can be obtained by a restricted type of variation of the action (3.1) of ref. 9:

$$S_m = \int d^4x_{\mathrm{I}}\, d^4x_{\mathrm{II}}\, \frac{1}{4} \begin{Bmatrix} (\partial_{\mathrm{I}}^{\dot b a}\, \chi_a^{\dot e})(\partial_{\mathrm{II}\dot e f}\, \chi_b^{f}) + (\partial_{\mathrm{II}}^{\dot b a}\, \psi_a^{\dot e})(\partial_{\mathrm{I}\dot e f} \psi_b^{f}) + \text{c.c.} \\ 2(\Phi_P - M_m^2)(\psi_d^{*\dot c}\, \chi_{\dot c}^{d} + \text{c.c.}) \end{Bmatrix} \tag{A4}$$

The quark coordinates have been transformed into a relative coordinate x between the quarks and a laboratory coordinate X for the meson according to (6.2) of ref. 8 or rather (A3) of ref. 13,

$$\begin{gathered} x^\mu = x_{\mathrm{II}}^\mu - x_{\mathrm{I}}^\mu, \qquad X^\mu = (1-a)x_{\mathrm{I}}^\mu - a x_{\mathrm{II}}^\mu \\ \partial_{\mathrm{I}}^{a\dot b} = \tfrac{1}{2}\partial_X^{a\dot b} - \partial^{a\dot b} = \tfrac{1}{2}(-\delta^{a\dot b}\partial_{X^0} - \vec\sigma^{a\dot b}\vec\partial_{\vec X}) + \delta^{a\dot b}\partial_0 + \vec\sigma^{a\dot b}\vec\partial \\ \partial_{\mathrm{II}\dot e f} = \tfrac{1}{2}\partial_{X\dot e f} + \partial_{\dot e f} = \tfrac{1}{2}(-\delta_{\dot e f}\partial_{X^0} + \vec\sigma_{\dot e f}\vec\partial_{\vec X}) - \delta_{\dot e f}\partial_0 + \vec\sigma_{\dot e f}\vec\partial \\ \partial_0 = \partial/\partial x^0, \qquad \vec\partial = \partial/\partial\vec x \end{gathered} \tag{A5}$$

Here, $a = 1/2 + \omega_0/E_0$ according to (6.6) of ref. 8 or (B1d) of ref. 13, ω_0 is the relative energy of the quarks, and E_0 is the meson energy. Symmetry between the quarks and antiquarks in π^0 as well as considerations below (4.1) of ref. 13 suggest that $\omega_0 = 0$, so that $a = 1/2$, which value has been used in (A5) and (2.3). Note also that $\int d^4X\, d^4x = \int d^4x_{\mathrm{I}}\, d^4x_{\mathrm{II}}$ according to (3.3) of ref. 9.

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